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Vibration Equation by Means of N-Fractional  
Calculus (Study on Applications for  
Fractional Calculus Operators in Univalent  
Function Theory)

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# Solutions to a Nearly Simple Harmonic Vibration Equation by Means of N- Fractional Calculus

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## Abstract

In this paper, solutions to a nearly simple harmonic vibration equation are discussed by means of N- fractional calculus, and some investigation of the solutions are reported.

**Keywords :** N- Fractional Calculus, Simple Harmonic Vibration Equation.

## Introduction ( Definition of Fractional Calculus )

( I ) Definition. ( by K. Nishimoto ) ( [ 1 ] Vol. 1 )

Let  $D = \{D_-, D_+\}$ ,  $C = \{C_-, C_+\}$ ,

$C_-$  be a curve along the cut joining two points  $z$  and  $-\infty + i\text{Im}(z)$ ,

$C_+$  be a curve along the cut joining two points  $z$  and  $\infty + i\text{Im}(z)$ ,

$D_-$  be a domain surrounded by  $C_-$ ,  $D_+$  be a domain surrounded by  $C_+$ .

( Here  $D$  contains the points over the curve  $C$  ).

Moreover, let  $f = f(z)$  be a regular function in  $D(z \in D)$ ,

$$f_v = (f)_v = {}_C(f)_v = \frac{\Gamma(v+1)}{2\pi i} \int_C \frac{f(\xi)}{(\xi-z)^{v+1}} d\xi \quad (v \notin \mathbb{Z}), \quad (1)$$

$$(f)_{-m} = \lim_{v \rightarrow -m} (f)_v \quad (m \in \mathbb{Z}^+), \quad (2)$$

where  $-\pi \leq \arg(\xi-z) \leq \pi$  for  $C_-$ ,  $0 \leq \arg(\xi-z) \leq 2\pi$  for  $C_+$ ,

$\xi \neq z$ ,  $z \in C$ ,  $v \in \mathbb{R}$ ,  $\Gamma$ ; Gamma function,

then  $(f)_v$  is the fractional differintegration of arbitrary order  $v$  ( derivatives of order  $v$  for  $v > 0$ , and integrals of order  $-v$  for  $v < 0$  ), with respect to  $z$ , of the function  $f$ , if  $|(f)_v| < \infty$ .

(II) On the fractional calculus operator  $N^v$  [ 3 ]

**Theorem A.** Let fractional calculus operator ( Nishimoto's Operator )  $N^v$  be

$$N^v = \left( \frac{\Gamma(v+1)}{2\pi i} \int_C \frac{d\xi}{(\xi-z)^{v+1}} \right) \quad (v \notin \mathbb{Z}) \quad [ \text{Refer to (1)} ], \quad (3)$$

with  $N^{-m} = \lim_{v \rightarrow -m} N^v \quad (m \in \mathbb{Z}^+), \quad (4)$

and define the binary operation  $\circ$  as

$$N^\beta \circ N^\alpha f = N^\beta N^\alpha f = N^\beta (N^\alpha f) \quad (\alpha, \beta \in \mathbb{R}), \quad (5)$$

then the set

$$\{ N^v \} = \{ N^v | v \in \mathbb{R} \} \quad (6)$$

is an Abelian product group ( having continuous index  $v$  ) which has the inverse transform operator  $(N^v)^{-1} = N^{-v}$  to the fractional calculus operator  $N^v$ , for the function  $f$  such that

$f \in F = \{ f; 0 \neq |f_v| < \infty, v \in \mathbb{R} \}$ , where  $f = f(z)$  and  $z \in C$ . ( vis.  $-\infty < v < \infty$  ).

(For our convenience, we call  $N^\beta \circ N^\alpha$  as product of  $N^\beta$  and  $N^\alpha$ .)

**Theorem B.** "F.O.G.  $\{N^\nu\}$ " is an "Action product group which has continuous index  $\nu$ " for the set of F. (F.O.G.; Fractional calculus operator group) [3].

### § 1. General solution to nearly simple harmonic vibration equations

**Theorem 1.** Let  $\varphi \in \mathcal{P}^\circ = \{\varphi : 0 \neq |\varphi_\nu| < \infty, \nu \in \mathbb{R}\}$ , then the homogeneous fractional order differintegral equation (nearly simple harmonic vibration equation for  $|\varepsilon| < 1$ )

$$\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0 \quad \begin{cases} \omega \neq 0, & \varphi = \varphi(t), \\ t, \varepsilon \in \mathbb{R}, & \varepsilon \neq -2 \end{cases} \quad (0)$$

has the following general solutions.

$$(i) \quad \varphi = \sum_{n=0}^m a_n e^{A(n, \varepsilon) \omega^{2/(2+\varepsilon)} t} \times [\cos B(n, \varepsilon) \omega^{2/(2+\varepsilon)} t + i \sin B(n, \varepsilon) \omega^{2/(2+\varepsilon)} t], \quad (1)$$

where

$$A(n, \varepsilon) = \cos \beta(n, \varepsilon) \quad (2), \quad B(n, \varepsilon) = \sin \beta(n, \varepsilon) \quad (3)$$

and

$$\beta(n, \varepsilon) = \pi(1+2n)/(2+\varepsilon) \quad (4)$$

for  $\varepsilon \in \mathbb{R}$ .

$$(ii) \quad \varphi = \sum_{n=0}^m a_n e^{G(n, \varepsilon) \omega^{r(\varepsilon)} t} \times [\cos H(n, \varepsilon) \omega^{r(\varepsilon)} t + i \sin H(n, \varepsilon) \omega^{r(\varepsilon)} t], \quad (5)$$

where

$$G(n, \varepsilon) = \cos \pi \left( \frac{1}{2} + n \right) r(\varepsilon) \quad (6), \quad H(n, \varepsilon) = \sin \pi \left( \frac{1}{2} + n \right) r(\varepsilon) \quad (7)$$

and

$$r(\varepsilon) = \sum_{k=0}^{\infty} (-\varepsilon/2)^k \quad (8)$$

for  $|\varepsilon| < 2$ .

$$(iii) \quad \varphi = \sum_{n=0}^m a_n e^{P(n, \varepsilon) \omega^{1-(\varepsilon/2)} t} \times [\cos Q(n, \varepsilon) \omega^{1-(\varepsilon/2)} t + i \sin Q(n, \varepsilon) \omega^{1-(\varepsilon/2)} t], \quad (9)$$

where

$$P(n, \varepsilon) = \cos \pi \left( \frac{1}{2} + n \right) \left( 1 - \frac{\varepsilon}{2} \right), \quad (10)$$

$$Q(n, \varepsilon) = \sin \pi \left( \frac{1}{2} + n \right) \left( 1 - \frac{\varepsilon}{2} \right), \quad (11)$$

for  $|\varepsilon| < 1$ .

Where  $a_n$  is an arbitrary constant correspond to  $\beta(n, \varepsilon)$  and

$m$  is finite when  $\varepsilon$  is a rational number, and

$m$  is infinite when  $\varepsilon$  is an irrational number.

**Note 1.** We must call (9) as approximate (or almost) general solution to equation (0), because it is not general solution in the strict sense.

**Proof of (i)**

Set

$$\varphi = e^{\lambda t}, \quad (12)$$

then operate  $N^{2+\varepsilon}$  to the both sides of (12), we have then

$$N^{2+\varepsilon} \varphi = \varphi_{2+\varepsilon} = \lambda^{2+\varepsilon} e^{\lambda t}. \quad [1] \quad (13)$$

Therefore, we have

$$\lambda^{2+\varepsilon} + \omega^2 = 0, \quad (14)$$

from (13), (12) and (0).

Hence

$$\lambda = (-\omega^2)^{1/(2+\varepsilon)} = e^{i\pi(1+2n)/(2+\varepsilon)} \omega^{2/(2+\varepsilon)} \quad (15)$$

$$= \gamma(n, \varepsilon) \quad (n = 0, 1, 2, \dots, m). \quad (16)$$

Then letting

$$\beta(n, \varepsilon) = \pi(1+2n)/(2+\varepsilon) \quad (4)$$

we have

$$\gamma(n, \varepsilon) = e^{i\beta(n, \varepsilon)} \omega^{2/(2+\varepsilon)} \quad (17)$$

$$= \{A(n, \varepsilon) + iB(n, \varepsilon)\} \omega^{2/(2+\varepsilon)} \quad (18)$$

where  $A(n, \varepsilon)$  and  $B(n, \varepsilon)$  are the ones shown by (2) and (3), respectively.

We have then a particular solution

$$\varphi = e^{\gamma(n, \varepsilon) t} \quad (19)$$

$$= e^{\{A(n, \varepsilon) + iB(n, \varepsilon)\} \omega^{2/(2+\varepsilon)} t} = \varphi|_{(n)} \quad (\text{denote}) \quad (20)$$

to equation (0).

Inversely (20) satisfies equation (0) clearly. Therefore, we have (1) from

$$\varphi = \sum_{n=0}^m a_n \cdot \varphi|_{(n)} \quad (21)$$

as the general solution to equation (0), where  $a_n$  is an arbitrary constant correspond to  $\beta(n, \varepsilon)$ .

**Proof of (ii)**

For  $|\varepsilon| < 2$ , we have

$$\frac{1}{2+\varepsilon} = \frac{1}{2} r(\varepsilon), \quad (22)$$

where

$$r(\varepsilon) = \sum_{k=0}^{\infty} (-\varepsilon/2)^k. \quad (8)$$

We have then

$$\beta(n, \varepsilon) = \pi(\frac{1}{2} + n) r(\varepsilon) \quad (23)$$

from (22) and (4).

Therefore, we have (5) from (23) and (1).

**Proof of (iii)**

For  $|\varepsilon| \ll 1$ , we have

$$r(\varepsilon) = \sum_{k=0}^{\infty} (-\varepsilon/2)^k \approx 1 - \frac{\varepsilon}{2}, \quad (24)$$

then

$$\beta(n, \varepsilon) = \pi(\frac{1}{2} + n)(1 - \frac{\varepsilon}{2}). \quad (25)$$

Therefore, we have (9) from (25) and (5).

## § 2. Investigation for $\varphi|_{(n)}$

Here we investigate the solutions  $\varphi|_{(n)}$  of the case (iii) in § 1.

**Theorem 2.** When  $\omega > 0$ ,

$$\varphi|_{(r)} \approx e^{(-1)^r \{(2r+1)\varepsilon\pi/4\}\omega t} [\cos Qt + (-1)^r i \sin Qt], \quad (26)$$

is convergent for

$$\begin{pmatrix} 0 < -\varepsilon \ll 1 & \text{for } r = 2k \\ 0 < \varepsilon \ll 1 & \text{for } r = 2k+1 \end{pmatrix}, \quad (27)$$

where

$$Q = \omega(1 - \frac{\varepsilon}{2} \log \omega), \quad (28)$$

$$\varphi|_{(n)} \approx e^{P(n, \varepsilon) \omega^{1-(\varepsilon/2)} t} [\cos Q(n, \varepsilon) \omega^{1-(\varepsilon/2)} t + i \sin Q(n, \varepsilon) \omega^{1-(\varepsilon/2)} t], \quad (29)$$

and  $k = 0, 1, 2, \dots$ .

**Proof. (I) Investigation for  $\varphi|_{(0)}$**

When  $|\varepsilon| \ll 1$ , we have (29) from § 1. (20) having § 1. (10) and (11), since

$$\varphi|_{(n)} \approx e^{\{P(n, \varepsilon) + i Q(n, \varepsilon)\} \omega^{1-(\varepsilon/2)} t}.$$

In the case of  $n = 0$ , we have

$$P(0, \varepsilon) \omega^{1-(\varepsilon/2)} \approx (\varepsilon \pi / 4) \omega, \quad (30)$$

and

$$Q(0, \varepsilon) \omega^{1-(\varepsilon/2)} \approx \omega(1 - \frac{\varepsilon}{2} \log \omega) = Q, \quad (31)$$

from (10) and (11) respectively, because we have

$$\cos \frac{\pi}{2} \left(1 - \frac{\varepsilon}{2}\right) = \sin \frac{\varepsilon \pi}{4} = \sum_{k=0}^{\infty} (-1)^k \frac{(\varepsilon \pi / 4)^{2k+1}}{(2k+1)!} \quad (|\varepsilon \pi / 4| < \infty), \quad (32)$$

$$\sin \frac{\pi}{2} \left(1 - \frac{\varepsilon}{2}\right) = \cos \frac{\varepsilon \pi}{4} = \sum_{k=0}^{\infty} (-1)^k \frac{(\varepsilon \pi / 4)^{2k}}{(2k)!} \quad (|\varepsilon \pi / 4| < \infty), \quad (33)$$

and

$$\omega^{1-(\varepsilon/2)} = \omega \cdot e^{\varepsilon \log \omega^{-1/2}} = \omega \sum_{k=0}^{\infty} \frac{(\varepsilon \log \omega^{-1/2})^k}{k!} \quad (|\varepsilon \log \omega^{-1/2}| < \infty). \quad (34)$$

Therefore, we have

$$\varphi|_{(0)} \approx e^{(\varepsilon \pi / 4) \omega t} [\cos Q t + i \sin Q t], \quad (|\varepsilon| < 1). \quad (35)$$

The solution (35) is divergent for  $\varepsilon > 0$  and is convergent (damping form) for  $\varepsilon < 0$  when  $\omega > 0$ . And we have

$$Q = \omega \left(1 - \frac{\varepsilon}{2} \log \omega\right) > \omega \quad (\varepsilon < 0, \omega > 1). \quad (36)$$

Then, letting

$$T_Q = 2\pi / Q; \text{ period of the function } \cos Q t,$$

and

$$T_\omega = 2\pi / \omega; \text{ period of the function } \cos \omega t,$$

we have

$$T_Q < T_\omega \quad (Q > \omega) \quad \text{when} \quad \varepsilon < 0.$$

That is, we have that "the period  $T_Q$  of  $\cos Q t = \cos \omega(1 - \frac{\varepsilon}{2} \log \omega)t$  is smaller than the one  $T_\omega$  of  $\cos \omega t$ " when  $\varepsilon < 0, \omega > 1$ .

### (II) Investigation for $\varphi|_{(1)}$

In the case of  $n = 1$ , we have

$$P(1, \varepsilon) \approx -(\varepsilon \pi / 4) \omega, \quad (37)$$

and

$$Q(1, \varepsilon) \approx -\omega(1 - \frac{\varepsilon}{2} \log \omega) = -Q, \quad (38)$$

from (10) and (11) respectively, because we have

$$\cos \frac{3\pi}{2} \left(1 - \frac{\varepsilon}{2}\right) = -\sin \frac{\varepsilon 3\pi}{4} = -\sum_{k=0}^{\infty} (-1)^k \frac{(\varepsilon 3\pi / 4)^{2k+1}}{(2k+1)!} \quad (|\varepsilon 3\pi / 4| < \infty), \quad (39)$$

$$\sin \frac{3\pi}{2} \left(1 - \frac{\varepsilon}{2}\right) = -\cos \frac{\varepsilon 3\pi}{4} = -\sum_{k=0}^{\infty} (-1)^k \frac{(\varepsilon 3\pi / 4)^{2k}}{(2k)!} \quad (|\varepsilon 3\pi / 4| < \infty), \quad (40)$$

and (34).

Therefore, we have

$$\varphi|_{(1)} \approx e^{-(\varepsilon 3\pi / 4) \omega t} [\cos Q t - i \sin Q t], \quad (|\varepsilon| < 1). \quad (41)$$

The solution (41) is convergent (damping form) for  $\varepsilon > 0$  and is divergent for  $\varepsilon < 0$  when  $\omega > 0$ . And we have

$$Q = \omega \left(1 - \frac{\varepsilon}{2} \log \omega\right) < \omega \quad (\varepsilon > 0, \omega > 1). \quad (42)$$

Then, in this case we have

$$T_Q > T_\omega \quad (Q < \omega) \quad \text{when} \quad \varepsilon > 0, \omega > 1.$$

That is, we have that " the period  $T_Q$  of  $\cos Qt = \cos \omega(1 - \frac{\varepsilon}{2} \log \omega)t$  is larger than the one  $T_\omega$  of  $\cos \omega t$  " when  $\varepsilon > 0, \omega > 1$ .

(III) Investigation for  $\varphi|_{(2)}$

In the case of  $n = 2$ , we have

$$P(2, \varepsilon) \approx (\varepsilon 5\pi / 4) \omega, \quad (43)$$

and

$$Q(2, \varepsilon) \approx \omega(1 - \frac{\varepsilon}{2} \log \omega) = Q, \quad (44)$$

from (10) and (11) respectively, because we have

$$\cos \frac{5\pi}{2} \left(1 - \frac{\varepsilon}{2}\right) = \sin \frac{\varepsilon 5\pi}{4} = \sum_{k=0}^{\infty} (-1)^k \frac{(\varepsilon 5\pi / 4)^{2k+1}}{(2k+1)!} \quad (|\varepsilon 5\pi / 4| < \infty), \quad (45)$$

$$\sin \frac{5\pi}{2} \left(1 - \frac{\varepsilon}{2}\right) = \cos \frac{\varepsilon 5\pi}{4} = \sum_{k=0}^{\infty} (-1)^k \frac{(\varepsilon 5\pi / 4)^{2k}}{(2k)!} \quad (|\varepsilon 5\pi / 4| < \infty), \quad (46)$$

and (34).

Therefore, we have

$$\varphi|_{(2)} \approx e^{(\varepsilon 5\pi / 4) \omega t} [\cos Qt + i \sin Qt], \quad (|\varepsilon| < 1). \quad (47)$$

The solution (47) is divergent for  $\varepsilon > 0$  and is convergent (damping form) for  $\varepsilon < 0, \omega > 0$ . And we have

$$Q = \omega(1 - \frac{\varepsilon}{2} \log \omega) > \omega \quad (\varepsilon < 0, \omega > 1). \quad (48)$$

Therefore we have

$$T_Q < T_\omega \quad (Q > \omega) \quad \text{when} \quad \varepsilon < 0, \omega > 1.$$

That is, we have that " the period  $T_Q$  of  $\cos Qt = \cos \omega(1 - \frac{\varepsilon}{2} \log \omega)t$  is smaller than the one  $T_\omega$  of  $\cos \omega t$  " when  $\varepsilon < 0, \omega > 1$ .

(IV) Repeating the same procedure as (I) ~ (III), we have this theorem clearly.

**Note.** Notice that when  $\omega > 0$ , the solutions

$\varphi|_{(2k)} \quad (k = 0, 1, 2, \dots)$  are convergent (damping form) for  $\varepsilon < 0$ , and

$\varphi|_{(2k+1)}$  are convergent (damping form) for  $\varepsilon > 0$ , respectively.

**Theorem 3.** When  $\omega > 0$  the nearly simple harmonic vibration equation

$$\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0 \quad \left( \begin{array}{l} \omega \neq 0, \quad \varphi = \varphi(t), \\ t, \varepsilon \in \mathbb{R}, \quad |\varepsilon| < 1 \end{array} \right) \quad (49)$$

has converging almost general solutions

$$\varphi \approx \sum_{k=0}^p a_{2k} \varphi|_{(2k)} \quad \text{when} \quad \varepsilon < 0 \quad (50)$$

and

$$\varphi \approx \sum_{k=0}^p a_{2k+1} \varphi|_{(2k+1)} \quad \text{when} \quad \varepsilon > 0, \quad (51)$$

where

$p$  is finite when  $\varepsilon$  is a rational number, and

$p$  is infinite when  $\varepsilon$  is an irrational number.

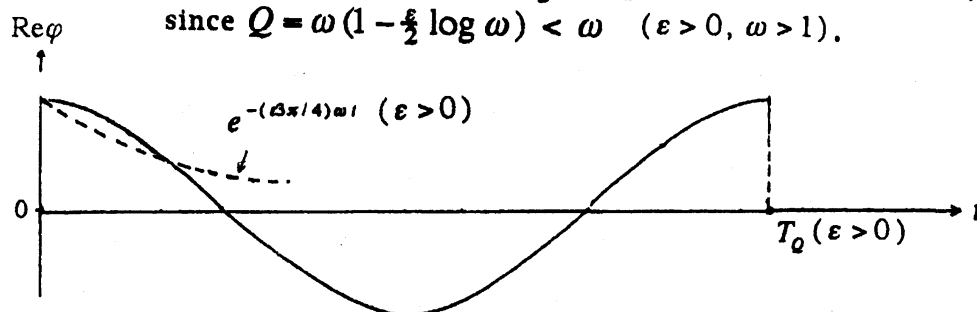
**Proof.** It is clear from the proof of Theorem 2, since we have (9) as solutions to equation (49)

### § 3. Some Graphs for $\varphi|_{(n)}$

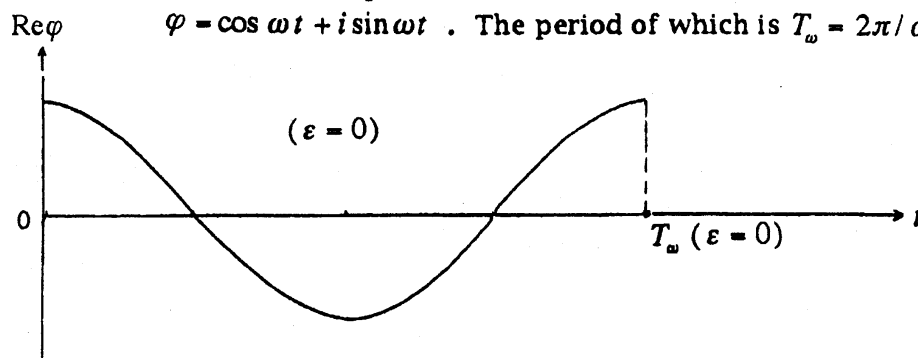
To equation  $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$  ( $0 < \varepsilon < 1$ ) we have a convergent (damping form) solution

$$\varphi|_{(1)} \approx e^{-(\varepsilon 3\pi/4)\omega t} [\cos Qt - i \sin Qt], \quad (\omega > 0).$$

To this function we have  $T_Q (> T_*)$  as its period for  $\varepsilon > 0$ , since  $Q = \omega(1 - \frac{\varepsilon}{2} \log \omega) < \omega$  ( $\varepsilon > 0, \omega > 1$ ).



To equation  $\varphi_2 + \varphi \cdot \omega^2 = 0$  ( $\varepsilon = 0$ ) we have the solution  $\varphi = \cos \omega t + i \sin \omega t$ . The period of which is  $T_\omega = 2\pi/\omega$ .



To equation  $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$  ( $0 < -\varepsilon < 1$ ) we have a convergent (damping form) solution

$$\varphi|_{(1)} \approx e^{(\varepsilon \pi/4)\omega t} [\cos Qt + i \sin Qt], \quad (\omega > 0).$$

To this function we have  $T_Q (< T_*)$  as its period for  $\varepsilon < 0$ , since  $Q = \omega(1 - \frac{\varepsilon}{2} \log \omega) > \omega$  ( $\varepsilon < 0, \omega > 1$ ).

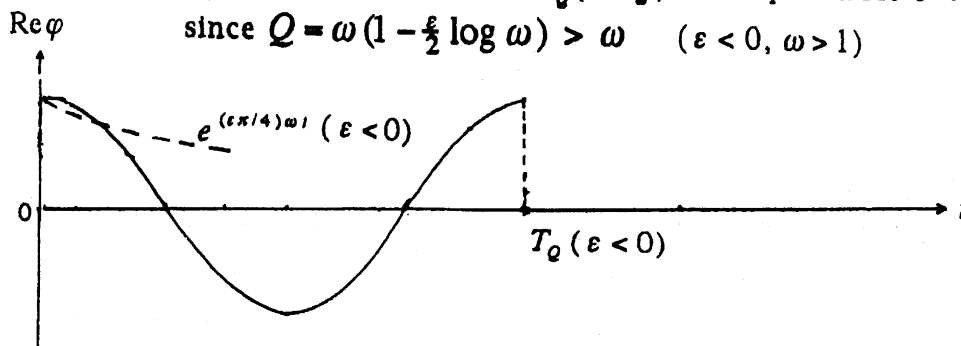


Fig. 1. Case of  $\omega > 1$ .



In Fig.1., the graphs of  $\text{Re } \varphi|_{(0)}$  and  $\text{Re } \varphi|_{(1)}$ , for the case  $\omega > 1$ , are shown in which the portion of amplitude and the one of vibration are separated. When  $0 < \omega < 1$ , we have  $T_Q(\text{of } \text{Re } \varphi|_{(1)}) < T_\omega$  ( $\varepsilon > 0$ ) and  $T_Q(\text{of } \text{Re } \varphi|_{(0)}) > T_\omega$  ( $\varepsilon < 0$ ). When  $\omega = 1$ , we have  $T_Q(\text{of } \text{Re } \varphi|_{(1)}) = T_\omega$  ( $\varepsilon > 0$ ) and  $T_Q(\text{of } \text{Re } \varphi|_{(0)}) = T_\omega$  ( $\varepsilon < 0$ ) since we have  $Q = \omega$  ( $\varepsilon > 0, \varepsilon < 0$ ). (See Fig.2 and Fig.3, respectively.)

(i) For example the nearly simple harmonic vibration equation

$$\varphi_{2+0.01} + \varphi \cdot \omega^2 = \frac{d^{2.01} \varphi}{dt^{2.01}} + \varphi \cdot \omega^2 = 0 \quad (\varepsilon = 0.01 > 0) \quad (52)$$

has solution

$$\varphi|_{(1)} \approx e^{-(0.01 \times 3\pi/4) \omega t} [\cos Qt - i \sin Qt] \quad (\omega > 0), \quad (53)$$

whose amplitude is  $e^{-(0.03\pi/4) \omega t}$  and the period is

$$T_Q = \frac{2\pi}{Q} = \frac{2\pi}{\omega(1-0.005 \cdot \log \omega)} > \frac{2\pi}{\omega} = T_\omega \quad (\omega > 1). \quad (54)$$

(ii) The equation

$$\varphi_{2-0.01} + \varphi \cdot \omega^2 = \frac{d^{1.99} \varphi}{dt^{1.99}} + \varphi \cdot \omega^2 = 0 \quad (\varepsilon = -0.01 < 0) \quad (55)$$

has solution

$$\varphi|_{(0)} \approx e^{(-0.01\pi/4) \omega t} [\cos Qt + i \sin Qt], \quad (\omega > 0), \quad (56)$$

whose amplitude is  $e^{(-0.01\pi/4) \omega t}$  and the period is

$$T_Q = \frac{2\pi}{Q} = \frac{2\pi}{\omega(1+0.005 \cdot \log \omega)} < \frac{2\pi}{\omega} = T_\omega \quad (\omega > 1). \quad (57)$$

(iii) The equation

$$\varphi_2 + \varphi \cdot \omega^2 = 0 \quad (58)$$

has solution

$$\varphi = \cos \omega t + i \sin \omega t. \quad (59)$$

This solution is produced in the process in which  $\varepsilon$  changes its sign in the equation

$$\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0 \quad (49)$$

Notice that; When  $\omega > 0$ ,

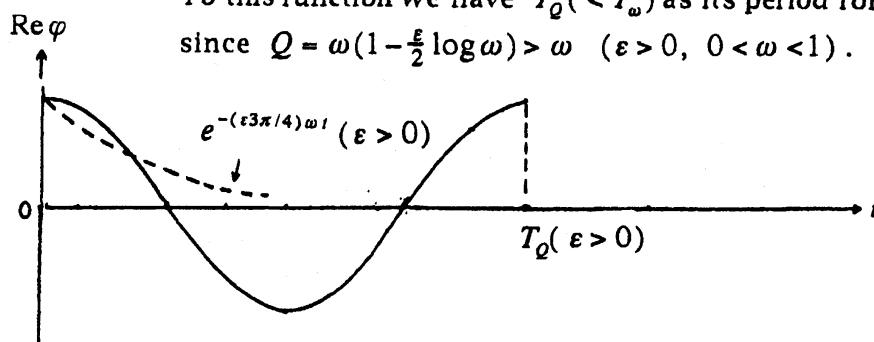
$\text{Re } \varphi|_{(n)}$ , having  $n = \text{even number}$ , give the same form damping vibration curves as the one of  $\text{Re } \varphi|_{(0)}$  for  $\varepsilon < 0$ , and

$\text{Re } \varphi|_{(n)}$ , having  $n = \text{odd number}$ , give the same form damping vibration curves as the one of  $\text{Re } \varphi|_{(1)}$  for  $\varepsilon > 0$ .

To equation  $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$  ( $0 < \varepsilon < 1$ ) we have a convergent (damping form) solution

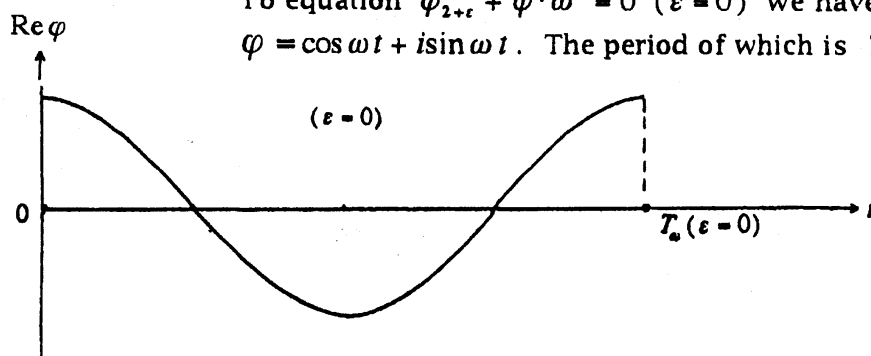
$$\varphi|_{(1)} \approx e^{-(\varepsilon 3\pi/4)\omega t} [\cos Qt - i \sin Qt], \quad (\omega > 0).$$

To this function we have  $T_Q (< T_\omega)$  as its period for  $\varepsilon > 0$ , since  $Q = \omega(1 - \frac{\varepsilon}{2} \log \omega) > \omega$  ( $\varepsilon > 0$ ,  $0 < \omega < 1$ ).



To equation  $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$  ( $\varepsilon = 0$ ) we have a solution

$$\varphi = \cos \omega t + i \sin \omega t. \quad \text{The period of which is } T_\omega = 2\pi / \omega.$$



To equation  $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$  ( $0 < -\varepsilon < 1$ ) we have a convergent (damping form) solution

$$\varphi|_{(0)} \approx e^{(\varepsilon \pi/4)\omega t} [\cos Qt + i \sin Qt], \quad (\omega > 0).$$

To this function we have  $T_Q (> T_\omega)$  as its period for  $\varepsilon < 0$ , since  $Q = \omega(1 - \frac{\varepsilon}{2} \log \omega) < \omega$  ( $\varepsilon < 0$ ,  $0 < \omega < 1$ ).

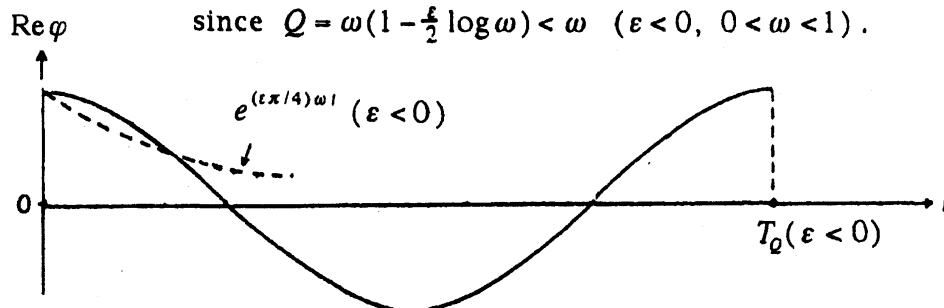
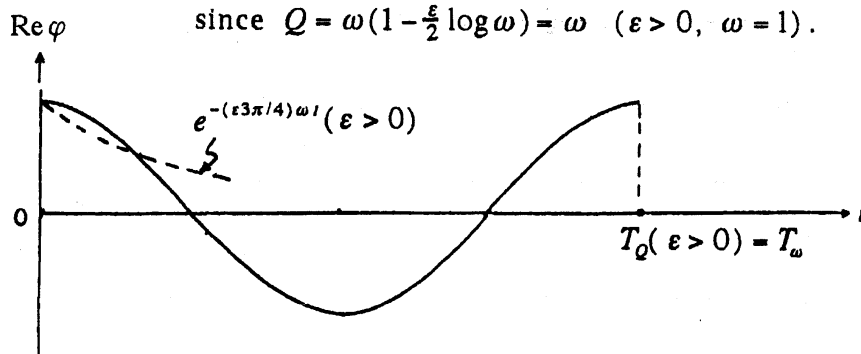


Fig. 2. Case of  $0 < \omega < 1$ .

To equation  $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$  ( $0 < \varepsilon < 1$ ) we have a convergent (damping form) solution

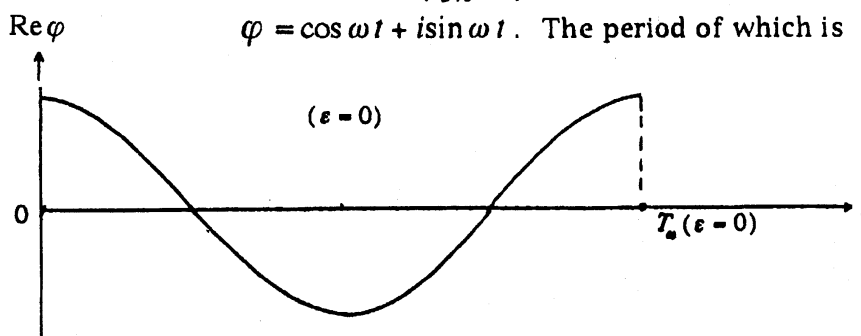
$$\varphi|_{(1)} \approx e^{-(\varepsilon 3\pi/4)\omega t} [\cos Qt - i \sin Qt], \quad (\omega > 0).$$

To this function we have  $T_Q = T_\omega$  as its period for  $\varepsilon > 0$ , since  $Q = \omega(1 - \frac{\varepsilon}{2} \log \omega) = \omega$  ( $\varepsilon > 0, \omega = 1$ ).



To equation  $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$  ( $\varepsilon = 0$ ) we have a solution

$$\varphi = \cos \omega t + i \sin \omega t. \quad \text{The period of which is } T_\omega = 2\pi / \omega.$$



To equation  $\varphi_{2+\varepsilon} + \varphi \cdot \omega^2 = 0$  ( $0 < -\varepsilon < 1$ ) we have a convergent (damping form) solution

$$\varphi|_{(0)} \approx e^{(\varepsilon \pi/4)\omega t} [\cos Qt + i \sin Qt], \quad (\omega > 0).$$

To this function we have  $T_Q = T_\omega$  as its period for  $\varepsilon < 0$ , since  $Q = \omega(1 - \frac{\varepsilon}{2} \log \omega) = \omega$  ( $\varepsilon < 0, \omega = 1$ ).

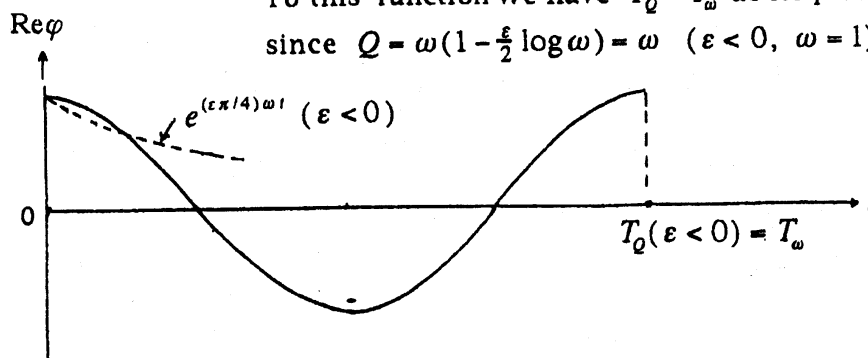


Fig. 3. Case of  $\omega = 1$ .

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